Diferencialne enačbe

1. red:

a) Poišči diferencialno enačbo katere rešitev je dana funkcija:

$$y = kx \qquad k \sim \text{parameter}$$

$$y' = k \Rightarrow y = y'x$$

b)

$$\frac{x^2}{a^2} + \frac{y^2}{4} = 1 \qquad a - \text{parameter}$$

$$\frac{1}{a^2}x^2 + \frac{1}{4}y^2 = 1 \quad /'$$

$$\frac{2x}{a^2} + \frac{\cancel{2}y'y}{\cancel{4}2} = \emptyset$$

$$\frac{2x}{a^2} + \frac{y'y}{2} = \emptyset \Rightarrow \frac{2x}{a^2} = -\frac{y'y}{2} \Rightarrow \frac{1}{a^2} = -\frac{y'y}{4x}$$

parameter

$$\Rightarrow -\frac{y'y}{4\cancel{x}} \cdot x^{\cancel{2}} + \frac{y^2}{4} = 1 \quad / \cdot 4$$

$$\boxed{-y'yx + y^2 = 4}$$

c)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a, b \sim \text{parameter}$$

$$\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1 \quad /'$$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = \emptyset$$

DE z ločljivimi spremenljivkami

x, y – lahko ločimo

$$g(y)\,dy = f(x)\,dx \quad / \int$$

$$\int g(y)\,dy = \int f(x)\,dx + C$$

$$y' = \frac{dy}{dx}$$

- Poišči splošno rešitev DE!

$$x y y' = 1 - x^2$$

$$y \frac{dy}{dx} = \frac{1-x^2}{x} \quad / \cdot dx$$

$$y\,dy = \frac{1-x^2}{x}\,dx \quad / \int$$

$$\int y\,dy = \int \left(\frac{1}{x} - x\right) dx + C$$

$$\frac{y^2}{2} = \ln x - \frac{x^2}{2} + C \quad / \cdot 2 \,/ \sqrt{\ }$$

$$y = \sqrt{2\ln x - x^2 + \underbrace{2C}_{D}}$$

- Poišči čisto rešitev, ki ustreza začetnim pogojem (Cauchyjeva naloga)

$$y' - 2x = 0, \qquad \text{ZP: } y(0) = 1$$

$$\frac{dy}{dx} = 2x \quad / \cdot dx$$

$$dy = 2x\,dx \quad / \int$$

$$y = \frac{\cancel{2}x^2}{\cancel{2}} = x^2 + C \sim \text{splošna rešitev}$$

$y(0) = 1$

$y = 1$
$x = 0$

$$1 = 0^2 + C \Rightarrow \boxed{C = 1}$$

Rešitev Cauchyjeve naloge:

$$y = x^2 + 1$$

LDE 1 reda

$$y' + a(x)y = f(x) \sim \text{splošna oblika}$$

– 1. način – variacija konstante

a) Rešimo homogeni del DE:

$$y' + a(x)y = 0$$

– Dobimo DE z ločljivimi spremenljivkami

$$\frac{dy}{dx} = -a(x)y$$

$$\frac{dy}{y} = -a(x)dx \quad /\int$$

$$\ln y = -\int a(x)dx + C$$

$$\Rightarrow \boxed{y_H = Ce^{-\int a(x)dx}}$$

b) Rešujemo nehomogeno enačbo z nastavkom

$$\underbrace{y_P = C(x)\cdot e^{-\int a(x)dx}}_{\text{partikularna}}$$

odvajamo

$$\underbrace{y_P' = C'(x)\cdot e^{-\int a(x)dx} + C(x)\cdot e^{-\int a(x)dx}\cdot(a(x))}_{\text{vstavimo v DE}}$$

$$\left(C'(x)\cdot e^{-\int a(x)dx} + \cancel{C(x)\cdot e^{-\int a(x)dx}\cdot(a(x))}\right) + \cancel{a(x)Ce^{-\int a(x)dx}} = f(x)$$

$$C'(x)\cdot e^{-\int a(x)dx} = f(x) \quad /: e^{-\int a(x)dx}$$

$$C'(x) = \frac{f(x)}{e^{-\int a(x)dx}} \quad /\int$$

$$\Rightarrow C(x) = \int \frac{f(x)}{e^{-\int a(x)dx}}$$

(5.)

$$x y' + y = \ln x + 1$$

a) homogeni del:

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$x\,dy = -y\,dx \quad /: xy$$

$$\frac{dy}{y} = -\frac{dx}{x} \quad /\int$$

$$\ln y = -\ln x + C$$

$$\ln y = \ln C - \ln x$$

$$y = \boxed{\frac{C}{x}} \sim y_H \sim \text{homogeni del rešitve DE}$$

vstavimo v DE

b) nehomogeni del:

$$y = \frac{C(x)}{x} \qquad C = C(x)$$

$$\underbrace{y' = \frac{C'(x)\cdot x - C(x)}{x^2} = \frac{C'x - C}{x^2}}_{\text{vstavimo v DE}} \sim y_p \sim \text{partikularni del}$$

$$\Rightarrow \cancel{x}\frac{C'x - \cancel{C}}{x^{\cancel{2}}} + \frac{\cancel{C}}{\cancel{x}} = \ln x + 1$$

$$C' = \ln x + 1 \quad /\int$$

$$C = \int(\ln x + 1)\,dx$$

$$C = (x\ln x - x) + x$$

$$C = x\ln x \sim \text{vstavimo nazaj v partikularno rešitev}$$

$$y_p = \frac{C(x)}{x} = \frac{x\ln x}{x} = \boxed{\ln x}$$

$$\int \ln x\,dx$$

Per partes $\int u\,dv = uv - \int v\,du$

$$u = \ln x \;/' \qquad dv = dx \;/\int$$

$$du = \frac{1}{x}dx \qquad v = x$$

$$\Rightarrow \ln x\cdot x - \int x\frac{1}{x}dx = x\ln x - x + C$$

Splošna rešitev:

$$y = y_H + y_p$$

$$\boxed{y = \frac{C}{x} + \ln x}$$

(6.)

LDE 1 reda

2. način – INTEGRACIJSKI MNOŽITELJ

obavezno!

$$1 \cdot y' + a(x)\, y = f(x)$$ ~ općenita oblika DE

1) $P(x) = \int a(x)\, dx$

2.) $\mu(x) = e^{P(x)}$

3) pomnožimo $f(x)$ s $\mu(x)$

DOBIMO $\Rightarrow$

$$\underbrace{\mu(x)\, y' + a(x)\, y\, \mu(x)} = f(x)\, \mu(x)$$

$$\left[y\, \mu(x) \right]' = \mu(x)\, f(x) \quad / \int$$

$$y\, \mu(x) = \int f(x)\, \mu(x)\, dx + C \quad / : \mu(x)$$

$$y = \frac{1}{\mu(x)} \int f(x)\, \mu(x)\, dx + \frac{1}{\mu(x)} \cdot C$$

PRIMERI $\quad x y' + y = \ln x + 1$

$$y' + \frac{y}{x} = \frac{\ln x + 1}{x}$$

$$y' + \underbrace{\frac{1}{x}}_{a(x)} y = \frac{\ln x + 1}{x}$$

$$a(x) = \frac{1}{x} \Rightarrow P(x) = \int a(x)\, dx = \int \frac{1}{x}\, dx = \boxed{\ln x}$$

$$\mu(x) = e^{P(x)} = e^{\ln x} = \boxed{x}$$

$$x y' + y = \ln x + 1$$

$$\left[x y \right]' = \ln x + 1 \quad / \int$$

$$y x = x \ln x - x + x + C$$

$$\boxed{y = \ln x + \frac{C}{x}}$$

7.)

$$xy' + y = -xy^2 \quad /:y^2$$

$$x\frac{y'}{y^2} + \frac{1}{y} = -x$$

Uvedba nove spremenljivke: $z = \frac{1}{y} = y^{-1}$

$$z' = -y^{-2}y' = \frac{-y'}{y^2}$$

$$x(-z') + z = x \quad /:(-x)$$

$$z' - \frac{1}{x}z = -1$$

$a(x)$

$$P(x) = \int -\frac{1}{x}dx = \boxed{-\ln x}$$

$$\mu(x) = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \boxed{\frac{1}{x}}$$ ~ pomnožimo (3.t.)

$\Downarrow$

$$\underbrace{\frac{1}{x}z' - \frac{1}{x^2}z}_{} = -\frac{1}{x}$$

$$\left[z\frac{1}{x}\right]' = -\frac{1}{x} \quad //\int$$

$$z\frac{1}{x} = -\ln x + C \quad /\cdot x$$

$$z = -x\ln x + Cx$$

$$\boxed{y = \frac{1}{-x\ln x + Cx}}$$

$z = \frac{1}{y} \Rightarrow y = \frac{1}{z}$

8.

NLDE n-tega reda s konstantnimi koeficienti

$$y^{(n)} + a_1(x) y^{(n-1)} + a_2(x) y^{(n-2)} + \dots + a_{n-1}(x) y' + a_n(x) y = f(x)$$

če so: $\left.\begin{array}{l} a_1(x) = a_1 \\ a_2(x) = a_2 \\ a_n(x) = a_n \end{array}\right\}$ konstante

$y'' + a_1 y' + a_2 y = f(x)$ ~ NLDE 2. reda s konstantnimi koeficienti

PRIMERI

1. $y'' + 2y' + y = \sin 2x$

$a_1 = 2$

$a_2 = 1$

$f(x) = \sin 2x$

$y'' - y = e^x$

$a_1 = 0$

$a_2 = -1$

$f(x) = e^x$

$$y'' + ay' + by = 0$$

$$\Downarrow$$

$$\lambda^2 + a\lambda^1 + b\lambda^0 = P(\lambda)$$

$\lambda^2 + a\lambda + b = 0$ karakteristična enačba

λ_1, λ_2 ~ lastne ali karakteristične enačbe

rešitve:

1) $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ = $y_H = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

2.) $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ = $y_H = (C_1 + C_2 x) e^{\lambda x}$

3.) kompleksni Ri $\lambda_1 = \overline{\lambda_2}$ = $y_H = e^{\alpha x}(C_1 \sin \beta x + C_2 \cos \beta x)$

$\lambda_1 = \alpha + i\beta$

$\lambda_2 = \alpha - i\beta$

9.

$$y'' + 5y' + 6y = 0$$

$$P(\lambda) = \lambda^2 + 5\lambda + 6$$

$$\lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda+2)(\lambda+3) = 0$$

$$\lambda_1 = -2 \qquad \lambda_1 \neq \lambda_2 \in \mathbb{R} \quad \Rightarrow \quad \boxed{y_H = C_1 e^{-2x} + C_2 e^{-3x}}$$
$$\lambda_2 = -3$$

$$y'' + 2y' + y = 0$$

$$P(\lambda) = \lambda^2 + 2\lambda + 1$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)(\lambda+1) = 0$$

$$\lambda_1 = \lambda_2 = -1 \in \mathbb{R} \quad \Rightarrow \boxed{y_H = (C_1 + C_2 x) \cdot e^{-x}}$$

$$y'' - 4y' + 13y = 0$$

$$P(\lambda) = \lambda^2 - 4\lambda + 13 = 0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{-36}}{2}$$

$$\lambda_{12} = 2 \pm j3$$

$$\alpha = 2 \qquad \Rightarrow \quad \boxed{y_H = e^{2x}\left(C_1 \sin 3x + C_2 \cos 3x\right)}$$
$$\beta = 3$$

Če določimo konstante dobimo <u>PARTIKULARNO</u> rešitev!

PRIMER

$$C_1 = 0 \qquad y = C_1 e^{-2x} + C_2 e^{-3x}$$
$$C_2 = 1$$

ali določimo začetne pogoje!

$$y(x_0) = y_0$$
$$y'(x_0) = y_0'$$

$$y(0) = 0$$
$$y'(0) = 1 \quad \Rightarrow \quad 0 = C_1 e^{-2\cdot 0} + C_2 e^{-3\cdot 0} = \underline{C_1 + C_2 = 0}$$

$$\downarrow$$

$$y' = -2C_1 e^{-2x} - 3C_2 e^{-3x} \quad \Rightarrow \quad 1 = -2C_1 e^{0} - 3C_2 e^{0}$$
$$\Rightarrow \quad \underline{-2C_1 - 3C_2 = 1}$$

Sistem:

$$\left.\begin{aligned} C_1 + C_2 &= 0 \quad /\cdot 2 \\ -2C_1 - 3C_2 &= 1 \end{aligned}\right\} +$$

$$\left.\begin{aligned} C_2 &= -1 \\ C_1 &= 1 \end{aligned}\right\} \text{vstavimo v DE}$$

$$\Downarrow$$

$$\underline{y = e^{-2x} - e^{-3x}}$$

partikularna rešitev, ki zadošča začetnim pogojem!

$$\boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}}$$

(11)

NEHOMOGENA LDE:

$$y'' + ay' + by = f(x)$$

1.) Nehomogeni določimo pripadajočo homogeno enačbo

2.) Rešimo in določimo splošno rešitev pripadajoče homogene enačbe

1.1.) $y'' + ay' + by = \varnothing$

2.1.) y_H

3.) Splošna rešitev nehomogene: $\boxed{y = y_H + y_P}$ ~ velja za n-ti red!

ISKANJE PARTIKULARNE REŠITVE

1.) Metoda nedoločenih koeficientov (z nastavki) (lažja)

2.) Metoda variacije konstant

1.) $y'' + y' - 2y = f(x)$ $\quad f(x) = e^{3x}$

$$y'' + y' - 2y = \varnothing$$

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda + 2)(\lambda - 1) = \varnothing$$

$$\lambda_1 = -2$$
$$\lambda_2 = 1$$

$$\lambda_1 \neq \lambda_2 \in \mathbb{R} \Rightarrow \underbrace{y = C_1 e^{-2x} + C_2 e^{x}}_{\text{Splošna rešitev pripadajoče homogene}} \sim y_H$$

nastavek: $y_P = A \cdot e^{3x}$ potrebujemo odvode za DE

$$\left.\begin{aligned} y' &= 3A \cdot e^{3x} \\ y'' &= 9A \cdot e^{3x} \end{aligned}\right\} \text{ vstavimo v } \quad y_P'' + y_P' - 2y_P = e^{3x}$$

$$9A \cdot e^{3x} + 3A \cdot e^{3x} - 2e^{3x} = e^{3x} \quad / : e^{3x}$$

$$A(9 + 3 - 2) = 1$$

$$A = \frac{1}{10}$$ ~ ena izmed partikularnih rešitev nehomogene

$y = y_H + y_P$

$$\boxed{y = C_1 e^{-2x} + C_2 e^{x} + \frac{1}{10} e^{3x}}$$

splošna rešitev nehomogene DE

LAPLACEOVA TRANSFORMACIJA (LT)

$$f(t) \xrightarrow[\text{NOVO FUNKCIJO}]{\text{PRIREDIMO}} F(s) \qquad F = \mathcal{L}(f)$$

Laplaceova transformiranka

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$f(t) \longrightarrow F(s)$$

$$1 \xrightarrow{LT} \frac{1}{s}$$

$$\mathcal{L}(y) = Y$$

$$\mathcal{L}(y)(s) = Y(s)$$

$$\mathcal{L}(f) = \mathcal{L}(f)$$

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2 \mathcal{L}(f) - s f(0) - f'(0)$$

PRIMER

$$y'' - y = -t \qquad y(0) = \emptyset \qquad y'(0) = 1$$

Klasičen način

- HOMOGENA

$$y'' - y = \emptyset$$

$$\lambda^2 - 1 = 0$$

$$\lambda_{1,2} = \pm 1$$

$$R: \; y_H = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$\boxed{y_H = C_1 e^{-t} + C_2 e^{t}}$$

- NEHOMOGENA (PARTIKULARNA)

$$y_P = Q_0(x) = At + B$$

$$y_P' = A$$

$$y_P'' = \emptyset$$

v DE

$$\Downarrow$$

$$0 - (At + B) = -t$$

$$-A = -1 \Rightarrow A = 1$$

$$B = 0$$

v y_P

$$\boxed{y_P = t}$$

$$\boxed{y_S = C_1 e^{-t} + C_2 e^{t} + t}$$

Pri pogojih!

$$y(0) = C_1 + C_2 = \emptyset$$

$$y_S' = -C_1 e^{-t} - C_2 e^{+t} + 1$$

$$y'(0) = -C_1 + C_2 + 1 = 1$$

SISTEM

$$\left.\begin{aligned} C_1 + C_2 &= \emptyset \\ C_2 = C_1 &= \emptyset \end{aligned}\right\} \text{vstavimo v } y_S$$

$$\boxed{y_S = t}$$